Dept. of Math./CMA Univ. of Oslo
Pure Mathematics
ISSN 0806–2439 October 2011

NON-LINEAR TIME-ADVANCED BACKWARD STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS WITH JUMPS

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Abstract. We prove an existence and uniqueness result for non-linear time-advanced backward stochastic partial differential equations with jumps (ABSPDEJs). We then apply our results to study a time-advanced backward type of stochastic generalized porous medium equations with jumps.

1. Introduction

The notion of backward stochastic differential equations (BSDEs) has received a lot of attention in the past two decades owing to a range of applications in stochastic optimal control theory, stochastic differential games, econometrics, mathematical finance, and non linear partial differential equation. See [8, 9, 15, 19, 32]. Since the work by Pardoux and Peng [24], there has been significant literature dedicated to the case of BSDE. See e.g., [1, 3, 9, 30].

Recently, Peng and Yang [25] introduced the notion of anticipated (or time-advanced) backward stochastic differential equations (ABSDEs). They proved existence and uniqueness of adapted solutions to ABSDEs under Lipschitz continuity of the drift. ABSDEs appear for example as adjoint processes when dealing with the maximum principle for stochastic control for a system with delay. See [6, 25, 20, 29]. The results in [25] were extended to the Poisson jumps case by Øksendal et al. [20] with an additional moving average type of delay in the drift coefficient.

In the present paper, we consider an infinite-dimensional version of the previous work. More exactly, we are interested in studying a class of time-advanced backward stochastic partial differential equations with jumps (ABSPDEJs) which includes the following

Date: First Version: July, 2011. This Version: October 2011.

2010 *Mathematics Subject Classification.* 60G51, 60H05, 60H15, 91G10.

Key words and phrases. Time-advanced BSDE, Lévy process, stochastic evolution equation.

The research leading to these results has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no [228087].

ABSPDEs:

$$
\begin{aligned}
dY(t,x) &= -A(t,Y(t,x))dt - E\Big[b(t,x)\Big|\mathcal{F}_t\Big]\mathrm{d}t + Z(t,x)\,\mathrm{d}B(t)\\
&\quad + \int_X Q(t,x,\zeta)\widetilde{N}(\mathrm{d}t,\mathrm{d}\zeta),\ \ (t,x)\in[0,T]\times\mathbb{R}^n
\end{aligned}
\tag{1.1}
$$

$$
Y(t,x)=G(t,x),\ Z(t,x)=G_1(t,x),\ Q(t,x,\zeta)=G_2(t,x,\zeta),\ \ t\in[T,T+\delta],
\tag{1.2}
$$

where $b(t,x) = b(t,x,Y(t,x),Y(t+\delta,x),Y_t(x),Z(t,x),Z(t+\delta,x),Z_t,Q(t,x,\cdot),Q(t+\delta,\cdot),Q_t(\cdot))$, $dY(t,x)$ denotes the Itô differential with respect to t and where A is a family of (nonlinear) operators satisfying some conditions (see Assumptiion A1), b satisfies Lipschitz continuous conditions (see Assumption A2) and Y_t is defined in (2.3).

We assume that $G(t,\omega)$ is a continuous H-valued $\mathcal{F}_t$-measurable process, $G_1(t,\omega)$ is a continuous $L_2(K,H)$-valued $\mathcal{F}_t$-measurable process, $G(t,\omega)$ is a continuous $L^2(\nu)$-valued $\mathcal{F}_t$-measurable process and for all $(t,x)\in[T,T+\delta]\times\mathbb{R}^n$ we have

$$
\begin{aligned}
&E\Big[\int_T^{T+\delta}\int_{\mathbb{R}^n}\|G(t,x)\|^2\,\mathrm{d}x\,\mathrm{d}t\Big]<\infty,\\
&E\Big[\int_T^{T+\delta}\int_{\mathbb{R}^n}\|G_1(t,x)\|^2\,\mathrm{d}x\,\mathrm{d}t\Big]<\infty,\\
&E\Big[\int_T^{T+\delta}\int_{\mathbb{R}^n}\int_{\mathbb{R}_0}\|G_2(t,x,\zeta)\|^2\nu(\mathrm{d}\zeta)\,\mathrm{d}x\,\mathrm{d}t\Big]<\infty.
\end{aligned}
$$

The function $b:[0,T]\times\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\Omega\longrightarrow\mathbb{R}$ and the terminal value functions G,G_1,G_2 are given.

We remark that, in the infinite dimensional case, the existence and uniqueness of adapted solutions of BSDEs has also been studied by several authors in the case with no delay. See [10, 11, 12, 13, 17, 18, 31].

In the case with delay, Øksendal et al [21] derived an existence and uniqueness result in finite dimension when the operator A is linear. We aim at giving conditions on the operator A, function b and the terminal value functions, which contain as a special case, the corresponding results in the finite dimensional ABSDEs case ($A=0$ or A linear). We establish an existence and uniqueness result of a *"strong"* solution $(Y(t,x),Z(t,x),Q(t,x,\zeta))$ of the ABSPDEJs in an appropriate set, that is the probability space, the noise and the Poisson random measure are given.

We shall in the present paper prove the existence and uniqueness of solutions of Equations (1.1)-(1.2) in infinite dimension and when A is a non-linear operator. We shall employ the Galerkin approximation method (see e.g., [2, 5, 7, 14, 22, 23, 26]), which consists of looking at the ABSPDEJs (1.1)-(1.2) as a special case of time-advanced backward stochastic evolution for Hilbert space valued processes.

The second motivation of our paper is to apply our results to study time-advanced backward-type stochastic generalized porous medium equations with jumps (BSPMEJs).

Let us consider first the deterministic homogeneous Dirichlet problem of the *generalized porous medium equation* (or *Filtration equation*) in complete form and with delay.

$$\partial_t Y = \Delta\Phi(Y) + b(t, Y(t), Y^t) \text{ in } [0,T] \times \mathcal{O}, \tag{1.3}$$
$$Y(t,x) = \varphi(t,x) \text{ in } [-\delta, 0] \times \mathcal{O}, \tag{1.4}$$
$$Y(t,x) = 0 \text{ in } [0,T] \times \partial\mathcal{O}, \tag{1.5}$$

where $\mathcal{O}$ is a bounded open subset in $\mathbb{R}^n$, Δ is the usual Laplace operator, $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous, monotone increasing function which satisfies some properties which will be given in Section 4. For $Y(\cdot) : \mathbb{R}_+ \to \mathbb{R}$, Y^t will denote the function defined by $Y^t(s) = Y(t-s)$ for $s \in [0,\delta]$. If we assume for instance that the previous equation represents the gas flow through a porous medium then Y represents the density and b represents the mass force source in the medium.

Let Y be solution to the problem (1.3)-(1.5). Define $\widetilde{Y}$ as the time reversal of T i.e.,

$$\widetilde{Y}(t,x) = Y(T-t,x) \text{ for } t \leq T.$$

Then $\widetilde{Y}(t,x)$ solves the following time-advanced backward generalized porous medium equation.

$$\partial_t \widetilde{Y} = \Delta\Phi(\widetilde{Y}) + b(t, \widetilde{Y}(t), \widetilde{Y}_t) \text{ in } [0,T] \times \mathcal{O}, \tag{1.6}$$
$$\widetilde{Y}(t,x) = \widetilde{\varphi}(t,x) \text{ in } [T, T+\delta] \times \mathcal{O}, \tag{1.7}$$
$$\widetilde{Y}(t,x) = 0 \text{ in } [0,T] \times \partial\mathcal{O}, \tag{1.8}$$

We shall study the following type of time-advanced BSPMEJs with a given terminal condition.

$$\begin{aligned} dY(t,x) &= -\Delta\Phi(Y(t,x))dt - E\Big[b(t,x)\Big|\mathcal{F}_t\Big]\mathrm{d}t + Z(t,x)\,\mathrm{d}B(t) \\ &\quad + \int_X Q(t,x,\zeta)\widetilde{N}(\mathrm{d}t,\mathrm{d}\zeta), \quad (t,x) \in [0,T] \times \mathcal{O} \end{aligned} \tag{1.9}$$

$$Y(t,x) = G(t,x),\ Z(t,x) = G_1(t,x),\ Q(t,x,\zeta) = G_2(t,x,\zeta),\ (t,x) \in [T, T+\delta] \times \mathcal{O}, \tag{1.10}$$

where $b(t,x) = b(t,x,Y(t,x),Y(t+\delta,x),Y_t(x),Z(t,x),Z(t+\delta,x),Z_t,Q(t,x,\cdot),Q(t+\delta,\cdot),Q_t(\cdot))$.

Let us mention that in the stochastic framework, since the solution must be adapted to the filtration generated by Brownian motion and the Poisson random measure, we need to condition the anticipated terms with respect to the filtration.

In [5, 33], the authors studied the existence and uniqueness of a strong solution for a class of stochastic functional differential equations driven by Brownian motion. The time-advanced backward stochastic generalized porous medium equation can be seen as the inverse problem to determine the stochastic coefficients from the terminal values.

The paper is organized as follows: In Section 2, we give the framework needed to establish our results. Section 3 contains the main results of the paper, and in Section 4, we apply the results to study time-advanced BSPMEJs.

2. Framework

In this section, we introduce the setting in which we shall prove our main result for time-advanced backward stochastic partial differential equations with jumps .

Let V, H, K, be three real separable Hilbert spaces such that V is continuously, densely embedded in H, with

$$V \hookrightarrow H \equiv H^* \hookrightarrow V^*. \tag{2.1}$$

Here, V^* is the topological dual of V. We assume in particular that V and V^* are uniformly convex. Denote by $\|\cdot\|_V$, $\|\cdot\|_{V^*}$ and $|\cdot|$ the norm in V, V^* and H respectively, by $\langle\cdot\rangle$ the duality product between V, V^* and and by $(\cdot)$ the scalar product in H.

Let $A(t,\cdot) : V \longrightarrow V^*$ be a family of (nonlinear) operators, defined a.e.t and $p \geq 2$. Assume the following conditions:

Assumption A1.

(a1) Coercivity: there exist $\alpha > 0$, $\lambda \in \mathbb{R}$ and an $\mathcal{F}_t$-adapted process $f \in L^1([0,T] \times \Omega, dt \otimes dP)$ such that:

$$2\langle A(t,u),u\rangle \leq \lambda|u|_H^2 - \alpha\|u\|_V^p + f(t) \quad \text{for all } u \in V, \ \text{a.e. } t$$

(a2) Boundedness: there exists $\gamma > 0$ and an $\mathcal{F}_t$-adapted process $g \in L^{\frac{p}{p-1}}([0,T]\times\Omega, dt\otimes dP)$ such that

$$\|A(t,u)\|_{V^*} \leq \gamma\|u\|_V^{p-1} + g(t) \quad \text{for all } u \in V, \ \text{a.e. } t$$

(a3) Measurability:

$$t \in (0,T) \longmapsto A(t,u), \quad \text{is Lebesgue-measurable for all } u \in V, \ \text{a.e. } t$$

(a4) Weak monotonicity: there exists $\lambda > 0$ such that

$$2\langle A(t,u) - A(t,v), u - v\rangle \leq \lambda|u-v|_H^2 \quad \text{for all } u,v \in V, \ \text{a.e. } t$$

(a5) Hemicontinuity: The map

$$\theta \in \mathbb{R} \to \langle A(t,u+\theta v), w\rangle \in \mathbb{R}$$

is continuous for $u, v, w \in V$, a.e. t

Let us mention that, in general, the operator A is not necessarily bounded from H to H.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space and $\{B(t),\, t \geq 0\}$ be a cylindrical Brownian motion with covariance space K on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ i.e., for any $k \in K$, $\langle B(t),k\rangle$ is a real valued Brownian motion with $E\Big[\langle B(t),k\rangle^2\Big] = t|k|_K^2$.

Let $(X, \mathcal{B}(X))$ be a measurable space, where X is a topological vector space. Denote by $\eta(t)$ a Lévy process on X. Let $\nu(\mathrm{d}\zeta)$ the Lévy measure of η and $L^2(\nu)$ be the L^2-space of square integrable H-valued measurable functions associated with ν. Define $p(t)$ by $p(t) = \Delta\eta(t) = \eta(t) - \eta(t-)$. It follows from the property of the Poisson process that $p = (p(t), t \in D_p)$ is a stationary Poisson point process on X with characteristic measure ν. Let $N(\mathrm{d}t, \mathrm{d}\zeta) = N_p(\mathrm{d}t, \mathrm{d}\zeta)$ be the Poisson counting measure associated with the Lévy process. Then $N(\mathrm{d}t, \mathrm{d}\zeta)$ has the compensator $E[N(\mathrm{d}t, \mathrm{d}\zeta)] = \nu(\mathrm{d}\zeta)\,\mathrm{d}t$ on $(X, \mathcal{B}(X))$. Denote by $\widetilde{N}(\mathrm{d}t, \mathrm{d}\zeta) = N(\mathrm{d}t, \mathrm{d}\zeta) - \nu(\mathrm{d}\zeta)\,\mathrm{d}t$ the compensated Poisson random measure.

Let $\{\mathcal{F}_t\}_{t\in[0,T]}$, be the σ-algebras generated by $\{B(s),\ N(s, A),\ A \in \mathcal{B}(X),\ s \leq t\}$. Then for a cylindrical Brownian motion, it is known that the following representation holds:

$$B(t) = \sum_{i=0}^{\infty} \beta_i(t) k_i, \tag{2.2}$$

where $\{k_i\}_{i\geq 1}$ is an orthonormal basis of K and $\{\beta_i(t),\ \ t \geq 0\}$ are independent standard Brownian motions.

Let L be a Hilbert space, $\delta > 0,\ p \geq 0$ and $T > 0$. We denote by $M_L^p = M^p(0, T+\delta; L)$, the reflexive Banach space of L-valued processes $(Y(t))_{t\in[0,T+\delta]}$ measurable and satisfying:

(1) $Y(t)$ is $\mathcal{F}_t$-measurable a.e. in t where $\mathcal{F}_t = \mathcal{F}_T,\ t \in [T, T+\delta]$.
(2) $E\Big[\int_0^{T+\delta} \|Y(t)\|_L^p \,\mathrm{d}t\Big] < \infty$.

For $p = 2$, $M^2(0, T+\delta; L)$ is a Hilbert space equipped with the following scalar product

$$\langle Y, Z\rangle_{M_L^2} = E\Big[\int_0^{T+\delta} \langle Y(t), Z(t)\rangle_L \,\mathrm{d}t\Big]$$

The space $S^2 = S^2(0, T+\delta; L)$ is defined in a similar way with (2), replaced by

$$\|Y\|_{S^2} = E\Big[\sup_{0\leq t\leq T+\delta} \|Y(t)\|_L^2\Big]^{\frac{1}{2}} < \infty.$$

Let $L_2(K, H)$ denote the space of Hilbert-Schmidt operators acting from K into H. Then $L_2(K, H)$ is a separable Hilbert space with the inner product $\langle Y, Z\rangle_{L_2(K,H)} = \sum_{i=0}^{\infty}\langle Yk_i, Zk_i\rangle_H$. Let $\|\cdot\|_{L_2(K,H)}$ represent the corresponding (Hilbert-Schmidt) norm.

Given a stochastic process $Y(t) \in M^p(0, T+\delta; V) \cap S^2(0, T+\delta; H)$, we denoted by Y_t, the $M^p(0, \delta; V) \cap S^2(0, \delta; H)$-valued stochastic process by setting

$$Y_t(s)(\omega) = Y(t+s)(\omega); \quad s \in [0, \delta] \tag{2.3}$$

As mentioned in the introduction, the purpose of the paper is to establish existence and uniqueness results for a class of time-advanced BSPDEJ of the form

$$\begin{cases} \mathrm{d}Y(t) & = -E\Big[b(t,Y(t),Y(t+\delta),Y_t,Z(t),Z(t+\delta),Z_t,Q(t,\cdot),Q(t+\delta,\cdot),Q_t(\cdot))\Big|\mathcal{F}_t\Big]\mathrm{d}t \\ & \qquad -A(t,Y(t))dt+Z(t)\,\mathrm{d}B(t)\,+\,\int_X Q(t,\zeta)\widetilde{N}(\mathrm{d}t,\mathrm{d}\zeta),\;\; t\in[0,T] \\ Y(t) & = G(t),\;\; Z(t)=G_1(t),\;\; Q(t,\zeta)=G_2(t,\zeta),\;\; t\in[T,T+\delta], \end{cases} \tag{2.4}$$

where A is a nonlinear operator, $G(t,\omega)$ is a continuous H-valued $\mathcal{F}_t$-measurable process, $G_1(t,\omega)$ is a continuous $L_2(K,H)$-valued $\mathcal{F}_t$-measurable process, $G_2(t,\omega)$ is a continuous $L^2(\nu)$-valued $\mathcal{F}_t$-measurable process and $b:[0,T]\times H^3\times(L_2(K,H))^3\times(L^2(\nu))^3\times\Omega\longrightarrow H$ satisfies the following conditions:

Assumption A2.

(b1) $E\Big[\int_0^T|b(t,0,0,0,0,0,0,0,0,0)|_H^2\mathrm{d}t\Big]<\infty.$

(b2) $t\in[0,T]\longmapsto b(t,y,y_1,y_2,z,z_1,z_2,q,q_1,q_2,\omega)$ is Lebesgue-measurable.

(b3) Lipschitz condition: There exists a C such that

$$\begin{aligned} &|b(t,y,y_1,y_2,z,z_1,z_2,q,q_1,q_2)-b(t,\bar{y},\bar{y}_1,\bar{y}_2,\bar{z},\bar{z}_1,\bar{z}_2,\bar{q},\bar{q}_1,\bar{q}_2)|_H \\ &\leq C\Big(|y-\bar{y}|_H+|y_1-\bar{y}_1|_H+|y_2-\bar{y}_2|_{L^2(0,\delta;H)}+\|z-\bar{z}\|_{L^2(K,H)}+\|z_1-\bar{z}_1\|_{L^2(K,H)} \\ &+\|z_2-\bar{z}_2\|_{L^2(0,\delta;L^2(K,H))}+\|q-\bar{q}\|_{L^2(\nu)}+\|q_1-\bar{q}_1\|_{L^2(\nu)}+\|q_2-\bar{q}_2\|_{L^2(0,\delta;L^2(\nu))}\Big) \end{aligned}$$

3. Main results

In this section, we present the main results of the paper.

3.1. **Existence and uniqueness.**

Theorem 3.1. *Assume that the terminal values $G\in S^2(T,T+\delta;H)\cap M^p(T,T+\delta;V)$, $G_1\in M^2(T,T+\delta;L_2(K,H))$ and $G_2\in M^2(T,T+\delta;L^2(\nu))$. Moreover, assume that the conditions of Assumptions A1-A2 are fulfilled. Then there exists a unique $H\times L_2(K,H)\times L^2(\nu)$-valued progressively measurable process $(Y(t),Z(t),Q(t))$ solution of equation (2.4) in $M^2(0,T+\delta;H)\cap M^p(0,T+\delta;V)\times M^2(0,T+\delta;L_2(K,H))\times M^2(0,T+\delta;L^2(\nu))$.*

We shall first prove the uniqueness of the solution when such a solution exists. Then we shall prove the existence in several lemmas.

Proof of the uniqueness. Let $(Y(t),Z(t),Q(t))$ and $(\bar{Y}(t),\bar{Z}(t),\bar{Q}(t))$ in $M^2(T,T+\delta;H)\cap M^p(T,T+\delta;V)\times M^2(T,T+\delta;L_2(K,H))\times M^2(T,T+\delta;L^2(\nu))$ be two solutions of the time-advanced BSPDEJ (2.4).

By the Itô formula, we get

$$
\begin{aligned}
0 &= |G(T)-\bar{G}(T)|_H^2 = |Y(T)-\bar{Y}(T)|_H^2 \\
&= |Y(t)-\bar{Y}(t)|_H^2 - 2\int_t^T \langle A(s,Y(s)) - A(s,\bar{Y}(s)), Y(s)-\bar{Y}(s)\rangle \mathrm{d}s \\
&- 2\int_t^T \Big(E\Big[b(s)-\bar{b}(s)\Big|\mathcal{F}_s\Big], Y(s)-\bar{Y}(s)\Big)\mathrm{d}s + \int_t^T \|Z(s)-\bar{Z}(s)\|_{L^2(K,H)}^2 \mathrm{d}s \\
&+ 2\int_t^T \Big(Y(s)-\bar{Y}(s), (Z(s)-\bar{Z}(s))\,\mathrm{d}B(s)\Big) + \int_t^T\int_X |Q(s,\zeta)-\bar{Q}(s,\zeta)|_H^2 N(\mathrm{d}s,\mathrm{d}\zeta) \\
&+ 2\int_t^T \Big(Y(s)-\bar{Y}(s), (Q(s,\zeta)-\bar{Q}(s,\zeta))\widetilde{N}(\mathrm{d}s,\mathrm{d}\zeta)\Big),
\end{aligned}
\tag{3.1}
$$

where we have used the short hand notation

$$
b(t) = b(t, Y(t), Y(t+\delta), Y_t, Z(t), Z(t+\delta), Z_t, Q(t,\cdot), Q(t+\delta,\cdot), Q_t(\cdot))
$$

This implies that

$$
\begin{aligned}
&|Y(t)-\bar{Y}(t)|_H^2 + \int_t^T \|Z(s)-\bar{Z}(s)\|_{L^2(K,H)}^2 \mathrm{d}s + \int_t^T\int_X |Q(s,\zeta)-\bar{Q}(s,\zeta)|_H^2 N(\mathrm{d}s,\mathrm{d}\zeta) \\
&= +2\int_t^T \langle A(s,Y(s)) - A(s,\bar{Y}(s)), Y(s)-\bar{Y}(s)\rangle \mathrm{d}s + 2\int_t^T \Big(E\Big[b(s)-\bar{b}(s)\Big|\mathcal{F}_s\Big], Y(s)-\bar{Y}(s)\Big)\mathrm{d}s \\
&- 2\int_t^T \Big(Y(s)-\bar{Y}(s), (Z(s)-\bar{Z}(s))\,\mathrm{d}B(s)\Big) - 2\int_t^T \Big(Y(s)-\bar{Y}(s), (Q(s,\zeta)-\bar{Q}(s,\zeta))\widetilde{N}(\mathrm{d}s,\mathrm{d}\zeta)\Big).
\end{aligned}
$$

Using weak monotonicity (a(2)), Lipschitz condition (b(3)) and taking the expectation, we get

$$
\begin{aligned}
&E\Big[|Y(t)-\bar{Y}(t)|_H^2\Big]+E\Big[\int_t^T \|Z(s)-\bar{Z}(s)\|_{L^2(K,H)}^2\mathrm{d}s\Big]\\
&+E\Big[\int_t^T\int_X |Q(s,\zeta)-\bar{Q}(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]\\
&\leq \lambda E\Big[\int_t^T |Y(s)-\bar{Y}(s)|_H^2 ds\Big]+\frac{1}{\varepsilon}E\Big[\int_t^T |Y(s)-\bar{Y}(s)|_H^2\mathrm{d}s\Big]\\
&+C_\varepsilon E\Big[\int_t^T |Y(s)-\bar{Y}(s)|_H^2\mathrm{d}s\Big]+C_\varepsilon E\Big[\int_t^T |Y_1(s)-\bar{Y}_1(s)|_H^2\mathrm{d}s\Big]\\
&+C_\varepsilon E\Big[\int_t^T \|Y_2(s)-\bar{Y}_2(s)\|_{L^2(0,\delta;H)}^2\mathrm{d}s\Big]+C_\varepsilon E\Big[\int_t^T \|Z(s)-\bar{Z}(s)\|_{L_2(K,H)}^2\mathrm{d}s\Big]\\
&+C_\varepsilon E\Big[\int_t^T \|Z_1(s)-\bar{Z}_1(s)\|_{L_2(K,H)}^2\mathrm{d}s\Big]+C_\varepsilon E\Big[\int_t^T \|Z_2(s)-\bar{Z}_2(s)\|_{L^2(0,\delta;L_2(K,H))}^2\mathrm{d}s\Big]\\
&+C_\varepsilon E\Big[\int_t^T\int_X |Q(s,\zeta)-\bar{Q}(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]+C_\varepsilon E\Big[\int_t^T\int_X |Q_1(s,\zeta)-\bar{Q}_1(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]\\
&+C_\varepsilon E\Big[\int_t^T\int_X \|Q_2(s,\zeta)-\bar{Q}_2(s,\zeta)\|_{L^2(0,\delta;H)}^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]. \qquad (3.2)
\end{aligned}
$$

Note also that since $Y(t)=\bar{Y}(t)=G(t)$ for $t\in[T,T+\delta]$, we have

$$
\begin{aligned}
E\Big[\int_t^T |Y_1(s)-\bar{Y}_1(s)|_H^2\mathrm{d}s\Big] &= E\Big[\int_t^T |Y(s+\delta)-\bar{Y}(s+\delta)|_H^2\mathrm{d}s\Big]\\
&\leq E\Big[\int_t^T |Y(s)-\bar{Y}(s)|_H^2\mathrm{d}s\Big], \qquad (3.3)
\end{aligned}
$$

and by interchanging the order of integration, we get

$$
\begin{aligned}
E\Big[\int_t^T |Y_2(s)-\bar{Y}_2(s)|_{L^2(0,\delta;H)}^2\mathrm{d}s\Big] &= E\Big[\int_t^T\Big(\int_0^\delta |Y(s+r)-\bar{Y}(s+r)|_H^2\mathrm{d}r\Big)\mathrm{d}s\Big]\\
&\leq E\Big[\int_t^{T+\delta} |Y(u)-\bar{Y}(u)|_H^2\mathrm{d}u\int_{u-\delta}^u \mathrm{d}s\Big]\\
&\leq \delta E\Big[\int_t^T |Y(s)-\bar{Y}(s)|_H^2\mathrm{d}s\Big]. \qquad (3.4)
\end{aligned}
$$

The same inequalities also hold for $Z-\bar{Z}$ and $Q-\bar{Q}$. Using (3.3) and (3.4), it follows from (3.2) that

$$
\begin{aligned}
&E\Big[|Y(t)-\bar{Y}(t)|_H^2\Big]+E\Big[\int_t^T\|Z(s)-\bar{Z}(s)\|_{L_2(K,H)}^2\mathrm{d}s\Big]\\
&+E\Big[\int_t^T\int_X|Q(s,\zeta)-\bar{Q}(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]\\
&\le(\lambda+\frac{1}{\varepsilon}+C_{\varepsilon,\delta})E\Big[\int_t^T|Y(s)-\bar{Y}(s)|_H^2\mathrm{d}s\Big]+C_{\varepsilon,\delta}^1E\Big[\int_t^T\|Y_2(s)-\bar{Y}_2(s)\|_{L^2(0,\delta;H)}^2\mathrm{d}s\Big]\\
&+C_{\varepsilon,\delta}^2E\Big[\int_t^T\int_X|Q(s,\zeta)-\bar{Q}(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]. \qquad (3.5)
\end{aligned}
$$

Now choosing ε small enough such that $C_{\varepsilon,\delta}^1<1$ and $C_{\varepsilon,\delta}^2<1$, we get

$$
E\Big[|Y(t)-\bar{Y}(t)|_H^2\Big]\le C_{\varepsilon,\delta,\lambda}E\Big[\int_t^T|Y(s)-\bar{Y}(s)|_H^2\mathrm{d}s\Big], \qquad (3.6)
$$

where $C_{\varepsilon,\delta,\lambda}=\lambda+\dfrac{1}{\varepsilon}+C_{\varepsilon,\delta}$. Hence, Gronwall's lemma obviously implies uniqueness.

□

Proof of the existence. We shall first give the following result on existence and uniqueness of a stochastic evolution equation in finite dimension.

Proposition 3.2. *Assume that $V=H=V^*=\mathbb{R}^d$ and $b=0$, and the operator A in (2.4) satisfies Assumption A1 with $\lambda=0$ in (a4). Then for $G\in S^2(T,T+\delta;H)\cap M^p(T,T+\delta;V)$, $G_1\in M^2(T,T+\delta;L_2(K,H))$ and $G_2\in M^2(T,T+\delta;L^2(\nu))$, there exists a unique $H\times L_2(K,H)\times L^2(\nu)$-valued progressively measurable process $(Y(t),Z(t),Q(t))$ solution of equation (2.4) in $M^2(0,T+\delta;H)\cap M^p(0,T+\delta;V)\times M^2(0,T+\delta;L_2(K,H))\times M^2(0,T+\delta;L^2(\nu))$.*

Proof. The result follows by combining the results in [4, 21, 33]. □

We shall prove the following lemmas.

Lemma 3.1. *Required conditions of Theorem 3.1. Moreover, assume that $b(t,y,y_1,y_2,z,z_1,z_2,q,q_1,q_2,\omega)=b(t,\omega)$ is independent of $y,y_1,y_2,z,z_1,z_2,q,q_1,q_2$ and that $E\Big[\int_0^T|b(t)|_H^2dt\Big]<\infty$. Then there exists a unique $H\times L_2(K,H)\times L^2(\nu)$-valued progressively measurable process $(Y(t),Z(t),Q(t))$ solution of equation (2.4) in $M^2(0,T+\delta;H)\cap M^p(0,T+\delta;V)\times M^2(0,T+\delta;L_2(K,H))\times M^2(0,T+\delta;L^2(\nu))$.*

Proof of Lemma 3.1.

The uniqueness has already been shown.

Existence :

Let $D(A)=\{v;\, v\in V,\, Av\in H\}$. Then the subspace $D(A)$ is dense in H. We fix an orthonormal basis $\{e_1,\dots,e_n,\dots\}$ of H where $e_i\in D(A)$ for all $i\geq 1$. Let $V_n=H_n=V_n^*$ be the vector space generated by $\{e_1,\dots,e_n\}$. Let $P_n\in\mathcal{L}(H,H_n=V_n)$ be the orthogonal projection from H into V_n. Then, P_n can be extended to an operator $\widetilde{P}_n$ from V^* onto $V_n^*=V_n$ as follows:

$$\widetilde{P}_n u=\sum_{i=1}^n\langle u,e_i\rangle e_i,\ \ u\in V^*.$$

Put $A_n=\widetilde{P}_nA$. Then A_n is an operator from V_n into $V_n=V_n^*$ satisfying Assumption A1.

Denote by K_n the subspace generated by $\{k_1,\dots,k_n\}$ with k_n given as in (2.2). Let $\bar{P}_n\in L(K,K_n)$ be the projection from K onto K_n. Let $B^n(t)$ be the K_n-valued Wiener process defined by $B^n(t)=\bar{P}_nB(t)$. Define

$$\mathcal{F}_t^n=\sigma\{B^n(s),\ N(s,A),\ A\in\mathcal{B}(X),\ s\leq t\}$$

completed by the probability measure P.

Define

$$b_n(t)=E\Big[P_nb(t)\Big|\mathcal{F}_t^n\Big]$$

and for $t\in[T,T+\delta]$,

$$G^n(t)=E\Big[P_nG(t)\Big|\mathcal{F}_T^n\Big],\ G_1^n(t)=E\Big[P_nG_1(t)\Big|\mathcal{F}_T^n\Big],\ G_2^n(t,\zeta)=E\Big[P_nG_2(t,\zeta)\Big|\mathcal{F}_T^n\Big].$$

Now, we consider the following time-advanced BSDEJ on V_n

$$\begin{cases}\mathrm{d}Y^n(t) &= A_n(t,Y^n(t))\mathrm{d}t+b_n(t)\mathrm{d}t+Z^n(t)\mathrm{d}B^n(t)\\ &\quad+\int_X Q^n(t,\zeta)\widetilde{N}(\mathrm{d}t,\mathrm{d}\zeta),\ \ t\in[0,T]\\ Y^n(t) &= G^n(t),\ Z^n(t)=G_1^n(t),\ Q^n(t,\zeta)=G_2^n(t,\zeta),\ \ t\in[T,T+\delta].\end{cases}\tag{3.7}$$

A_n is an operator satisfying Assumption A1 on the finite dimensional space V_n onto V_n. It is also easy to check that B^n,G^n,G_1^n and G_n^2 satisfy the assumptions of Proposition 3.2 by replacing V,H,V^* by V_n,H_n,V_n^*. We can then conclude that Equation (3.7) has a unique $\mathcal{F}_t^n$-adapted solution $(Y^n(t),Z^n(t),Q^n(t,\cdot))\in V_n\times L_2(K_n,V_n)\times L^2(\nu)$.

We also have that for each n and t

$$|G^n(T)|_H\leq|G(T)|_H,\ \ \lim_{n\to\infty}E\Big[|G^n(T)-G(T)|_H^2\Big]=0\tag{3.8}$$

$$|G^n(t)|_H\leq|G(t)|_H,\ \ \lim_{n\to\infty}E\Big[\int_T^{T+\delta}|G^n(t)-G(t)|_H^2\mathrm{d}t\Big]=0\tag{3.9}$$

$$\|G_1^n(t)\|_{L_2(H,K)}\leq\|G_1(t)\|_{L_2(H,K)},\ \ \lim_{n\to\infty}E\Big[\int_T^{T+\delta}\|G_1^n(t)-G_1(t)\|_{L_2(H,K)}^2\mathrm{d}t\Big]=0\tag{3.10}$$

$$\|G_2^n(t)\|_{L^2(\nu)}\leq\|G_2(t)\|_{L^2(\nu)},\ \ \lim_{n\to\infty}E\Big[\int_T^{T+\delta}\int_X|G_2^n(t,\zeta)-G_2(t,\zeta)\|_H\nu(\mathrm{d}\zeta)\mathrm{d}t\Big]=0\tag{3.11}$$

$$|b^n(t)|_H \le |b(t)|_H, \ \lim_{n\to\infty} E\Big[\int_0^T |b^n(t)-b(t)|_H^2 \mathrm{d}t\Big] = 0 \tag{3.12}$$

In what follows, we shall split the proof into three steps. In the first step, we shall show that the sequence $(Y^n(t), Z^n(t), Q^n(t,\cdot))$ is bounded in $M^2(0,T+\delta;H)\cap M^p(0,T+\delta;V)\times M^2(0,T+\delta;L_2(K,H))\times M^2(0,T+\delta;L^2(\nu))$. In step 2, we shall show that the weak limit as a version that satisfies the following time-advanced BSPDE

$$\begin{cases} dY(t) &= -b(t)\mathrm{d}t - X(t)\mathrm{d}t + Z(t)\,\mathrm{d}B(t) + \int_X Q(t,\zeta)\widetilde{N}(\mathrm{d}t,\mathrm{d}\zeta), \ \ t\in[0,T] \\ Y(t) &= G(t), \ Z(t) = G_1(t), \ Q(t,\zeta) = G_2(t,\zeta), \ \ t\in[T,T+\delta]. \end{cases}$$

In the last step, we shall prove that $X(t) = A(t,Y(t))$ in $M^2(0,T;V^*)$.

Step 1. Let show that the sequence $(Y^n(t), Z^n(t), Q^n(t,\cdot))$ is bounded in $M^2(0,T+\delta;H)\cap M^p(0,T+\delta;V)\times M^2(0,T+\delta;L_2(K,H))\times M^2(0,T+\delta;L^2(\nu))$. By the Itô formula, we have

$$\begin{aligned} E\Big[|Y^n(t)|_H^2\Big] &= E\Big[|G^n(T)|_H^2\Big] + E\Big[2\int_t^T \langle \widetilde{P}_n A(s,Y^n(s)), Y^n(s)\rangle \mathrm{d}s\Big] \\ &\quad + 2E\Big[\int_t^T \Big(b_n(s), Y(s)\Big)\mathrm{d}s\Big] - E\Big[\int_t^T \|Z^n(s)\|_{L_2(K_n,V_n)}^2 \mathrm{d}s\Big] \\ &\quad - E\Big[\int_t^T\int_X |Q^n(s,\zeta)|_H^2 \nu(\mathrm{d}\zeta)\mathrm{d}t\Big], \end{aligned} \tag{3.13}$$

where $\|Z^n(s)\|_{L_2(K_n,V_n)}^2 = \sum_{i,j=1}^n \Big(Z^n_{(i,j)}(s)\Big)^2$ denotes the Hilbert-Schmidt norm. Using the coercivity argument, we obtain

$$\begin{aligned} E\Big[|Y^n(t)|_H^2\Big] &\le E\Big[|G(T)|_H^2\Big] - \alpha E\Big[\int_t^T \|Y^n(s)\|_V^p \mathrm{d}s\Big] + \lambda E\Big[\int_t^T |Y^n(s)|_H^2 \mathrm{d}s\Big] \\ &\quad + E\Big[\int_t^T f(s)\mathrm{d}s\Big] + \frac{1}{\varepsilon}E\Big[\int_t^T |Y^n(s)|_H^2\mathrm{d}s\Big] + C_\varepsilon E\Big[\int_t^T |b(s)|_H^2\mathrm{d}s\Big] \\ &\quad - E\Big[\int_t^T \|\bar{Z}^n(s)\|_{L_2(K,H)}^2 \mathrm{d}s\Big] - E\Big[\int_t^T\int_X |Q^n(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}t\Big], \end{aligned} \tag{3.14}$$

where $\bar{Z}^n(s) = \bar{P}_n Z^n(s)$ with $\bar{P}_n$ been the projection from K into K_n. Therefore,

$$E\Big[|Y^n(t)|_H^2\Big] \le (\lambda + \frac{1}{\varepsilon})E\Big[\int_t^T |Y^n(s)|_H^2\mathrm{d}s\Big] + E\Big[|G(T)|_H^2 + C_\varepsilon\int_t^T |b(s)|_H^2\mathrm{d}s + \int_t^T |f(s)|\mathrm{d}s\Big]$$

It follows from Gronwall's lemma that

$$E\Big[\int_0^T |Y^n(s)|_H^2\mathrm{d}s\Big] \le CE\Big[|G(T)|_H^2 + \int_0^T |b(s)|_H^2\mathrm{d}s + \int_0^T |f(s)|\mathrm{d}s\Big]$$

for a suitable constant C. Since the right hand side does not depend on n we can conclude that $(Y^n,\ n \geq 1)$ is bounded in $M^2(0, T+\delta; H)$. This also implies that the sequence $(Y^n(t), \bar{Z}^n(t), Q^n(t,\cdot))$ is bounded in $M^2(0,T+\delta;H) \cap M^p(0,T+\delta;V) \times M^2(0,T+\delta; L_2(K,H)) \times M^2(0,T+\delta;L^2(\nu))$. Moreover, it follows from boundedness of A (condition (a2)) that the sequence $(A(\cdot, Y^n),\ n \geq 1)$ is bounded in $M^{p'}(0,T;V^*)$ (where p' is the conjugate of p). Hence, by the weak compactness of Hilbert spaces and separable reflexive Banach spaces, there exist a subsequence $(Y^{n_k}(\cdot), \bar{Z}^{n_k}(\cdot), Q^{n_k}(\cdot), A(\cdot, Y^{n_k})\ k \geq 1)$ of $(Y^n(\cdot), Z^n(\cdot), Q^n(\cdot))$ such that

$$Y^{n_k} \to Y \text{ weakly in } M^p(0,T+\delta;V) \tag{3.15}$$

$$Y^{n_k}(0) \to Y_0 \text{ weakly in } L^2(\Omega;H) \tag{3.16}$$

$$\bar{Z}^{n_k} \to Z \text{ weakly in } M^2(0,T+\delta;L_2(K,H)) \tag{3.17}$$

$$Q^{n_k}(\cdot) \to Q \text{ weakly in } M^2(0,T+\delta;L^2(\nu)) \tag{3.18}$$

$$A(\cdot, Y^{n_k}) \to X \text{ weakly in } M^{p'}(0,T;V^*) \tag{3.19}$$

Step 2. We shall now show that (Y,Z,Q) has a version which is solution of the time-advanced BSDEJ (2.4). We first remark that for $n, i \geq 1$, we have

$$\begin{aligned}
\mathrm{d}(Y^n(t), e_i) &= \langle \bar{P}_n A(t, Y^n(t)), e_i\rangle \mathrm{d}t - (b_n(t), e_i)\mathrm{d}t + (\bar{Z}^n(t)\mathrm{d}B^n(t), e_i) \\
&\quad + \int_X (Q^n(s,\zeta), e_i)\widetilde{N}(\mathrm{d}t, \mathrm{d}\zeta) \\
&= -\langle A(t, Y^n(t)), e_i\rangle \mathrm{d}t - (b_n(t), e_i)\mathrm{d}t + (\bar{Z}^n(t)\mathrm{d}B(t), e_i) \\
&\quad + \int_X (Q^n(s,\zeta), e_i)\widetilde{N}(\mathrm{d}t, \mathrm{d}\zeta)
\end{aligned} \tag{3.20}$$

Let $h(t)$ be an absolutely continuous function from $\mathbb{R}$ to $\mathbb{R}$ with $h'(\cdot) \in L^([0,T])$.

If φ is a function from $[0,T]$ into $\mathbb{R}$, we define $\bar{\varphi}$ from $[-\delta, T+\delta]$ into $\mathbb{R}$ as follows:

$$\bar{\varphi}(t) = \begin{cases} \varphi(t), & \text{if } t \in [0,T], \\ 0, & \text{otherwise.} \end{cases}$$

The latter and Itô formula permit us to rewrite (3.20) as follows

$$\begin{aligned}
&(Y^n(T), e_i)h(T-t) - (Y^n(0), e_i)h(-t) \\
&= -\overline{\int_0^T h(s-t)\langle A(s, Y^n(s)), e_i\rangle \mathrm{d}s} - \overline{\int_0^T h(s-t)(b_n(s), e_i)\mathrm{d}s} \\
&\quad + \overline{\int_0^T h(s-t)\mathrm{d}(\int_0^s \bar{Z}^n(r)\mathrm{d}B(r), e_i)} + \overline{\int_0^T \int_X h(s-t)(Q^n(s,\zeta), e_i)\widetilde{N}(\mathrm{d}s, \mathrm{d}\zeta)} \\
&\quad + \overline{\int_0^T h'(s-t)(Y^n(s), e_i)\mathrm{d}s}, \quad \forall t \in [-\delta, T+\delta].
\end{aligned} \tag{3.21}$$

We use the fact that the linear maps

$$Z \mapsto \int_0^T h(s-t)\mathrm{d}(\int_0^s \bar{Z}^n(r)\mathrm{d}B^n(r), e_i) = \sum_{j=1}^{\infty} \int_0^T h(s-t)(Z(s,k_j), e_i)\mathrm{d}\beta^j(s)$$

is continuous from $M^2(0,T+\delta; L_2(K,H))$ into $L^2(\Omega)$, and that

$$Z \in M^2(0,T+\delta; L^2(\nu)) \mapsto \int_0^T \int_X h(s-t)(Q^n(s,\zeta), e_i)\widetilde{N}(\mathrm{d}s, \mathrm{d}\zeta)$$

is continuous from $M^2(0,T+\delta; L^2(\nu))$ into $L^2(\Omega)$. We replace n by n_k in (3.21) and take the weak limit in $L^2(\Omega)$ to obtain

$$\begin{aligned}
&(Y(T), e_i)h(T-t) - (Y_0, e_i)h(-t) \\
&= -\overline{\int_0^T h(s-t)\langle A(s,Y(s)), e_i\rangle \mathrm{d}s} - \overline{\int_0^T h(s-t)(b(s), e_i)\mathrm{d}s} \\
&+ \overline{\int_0^T h(s-t)\mathrm{d}(\int_0^s Z(r)\mathrm{d}B(r), e_i)} + \overline{\int_0^T \int_X h(s-t)(Q(s,\zeta), e_i)\widetilde{N}(\mathrm{d}s, \mathrm{d}\zeta)} \\
&+ \overline{\int_0^T h'(s-t)(Y(s), e_i)\mathrm{d}s}, \quad \forall t \in [-\delta, T+\delta].
\end{aligned} \tag{3.22}$$

Choose for $n \geq 1$

$$h_n(u) = \begin{cases} 1, & u \geq \frac{1}{2n} \\ 1 - \frac{1}{n}(s - \frac{1}{2n}), & -\frac{1}{2n} \leq u \leq \frac{1}{2n} \\ 0, & u \leq -\frac{1}{2n}. \end{cases}$$

We replace $h(\cdot)$ by $h_n(\cdot)$ in (3.22) to get

$$\begin{aligned}
&(Y(T), e_i)h_n(T-t) - (Y_0, e_i)h_n(-t) \\
&= -\overline{\int_0^T h_n(s-t)\langle A(s,Y(s)), e_i\rangle \mathrm{d}s} - \overline{\int_0^T h_n(s-t)(b(s), e_i)\mathrm{d}s} \\
&+ \overline{\int_0^T h_n(s-t)\mathrm{d}(\int_0^s Z(r)\mathrm{d}B(r), e_i)} + \overline{\int_0^T \int_X h_n(s-t)(Q(s,\zeta), e_i)\widetilde{N}(\mathrm{d}s, \mathrm{d}\zeta)} \\
&+ \overline{\int_{t-\frac{1}{2n}}^{t+\frac{1}{2n}} (Y(s), e_i)\mathrm{d}s}, \quad \forall t \in [-\delta, T+\delta].
\end{aligned} \tag{3.23}$$

We apply twice the change of variable and by letting n tend to infinity leads to

$$
\begin{aligned}
&(Y(T), e_i)h(T-t) - (Y_0, e_i)h(-t) \\
&= -\overline{\int_t^T \langle A(s, Y(s)), e_i\rangle \mathrm{d}s} - \overline{\int_t^T (b(s), e_i)\mathrm{d}s} + \overline{\int_t^T \mathrm{d}(\int_0^s Z(r)\mathrm{d}B(r), e_i)} \\
&+ \overline{\int_t^T \int_X (Q(s,\zeta), e_i)\widetilde{N}(\mathrm{d}s, \mathrm{d}\zeta)} + \overline{(Y(t), e_i)}, \quad \forall t \in [-\delta, T+\delta], \;\; i \geq 1,
\end{aligned} \tag{3.24}
$$

where h is defined from $\mathbb{R}$ into $\mathbb{R}$ by

$$
h(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{if } u < 0. \end{cases}
$$

It then follows that $Y(0) = Y_0$ and

$$
\begin{cases} dY(t) & = -b(t)\mathrm{d}t - X(t)\mathrm{d}t + Z(t)\,\mathrm{d}B(t) + \int_X Q(t,\zeta)\widetilde{N}(\mathrm{d}t, \mathrm{d}\zeta), \;\; t \in [0,T] \\ Y(t) & = G(t), \;\; Z(t) = G_1(t), \;\; Q(t,\zeta) = G_2(t,\zeta), \;\; t \in [T, T+\delta]. \end{cases} \tag{3.25}
$$

Step 3. We shall prove that $X(t) = A(t, Y(t))$ in $M^{p'}(0,T;V^*)$. It follows from (3.13) and the weak monotonicity argument (a4) that for any $\Theta \in M^p(0,T;V) \cap M^2(0,T;H)$

$$
\begin{aligned}
&E\Big[|Y^{n_k}(t)|_H^2\Big] + E\Big[\int_t^T \|\bar{Z}^{n_k}(s)\|^2_{L_2(K_n,V_n)}\mathrm{d}s\Big] + E\Big[\int_t^T \int_X |Q^{n_k}(s,\zeta)|_H^2 \nu(\mathrm{d}\zeta)\mathrm{d}t\Big] \\
&= E\Big[|G^{n_k}(T)|_H^2\Big] + 2E\Big[\int_t^T \langle A(s, Y^{n_k}(s)) - A(s, \Theta(s)), \Theta(s)\rangle \mathrm{d}s\Big] \\
&+ 2E\Big[\int_t^T \langle A(s, \Theta(s)), Y^{n_k}(s)\rangle \mathrm{d}s\Big] + 2E\Big[\int_t^T \langle A(s, Y^{n_k}(s)) - A(s, \Theta(s)), Y^{n_k}(s) - \Theta(s)\rangle \mathrm{d}s\Big] \\
&+ 2E\Big[\int_t^T \Big(b_{n_k}(s), Y(s)\Big)\mathrm{d}s\Big] \\
&\leq E\Big[|G^{n_k}(T)|_H^2\Big] + 2E\Big[\int_t^T \langle A(s, Y^{n_k}(s)) - A(s, \Theta(s)), \Theta(s)\rangle \mathrm{d}s\Big] \\
&+ 2E\Big[\int_t^T \langle A(s, \Theta(s)), Y^{n_k}(s)\rangle \mathrm{d}s\Big] + \lambda E\Big[\int_t^T |Y^{n_k}(s) - \Theta(s)|_H^2 \mathrm{d}s\Big] \\
&+ 2E\Big[\int_t^T \Big(b_{n_k}(s), Y^{n_k}(s)\Big)\mathrm{d}s\Big]
\end{aligned} \tag{3.26}
$$

217 We take the limit as k goes to infinity and set $t=0$ to get,

$$\liminf_{k\to\infty} E\Big[|Y^{n_k}(0)|_H^2\Big] + \liminf_{k\to\infty} E\Big[\int_0^T \|\bar{Z}^{n_k}(s)\|^2_{L_2(K_n,V_n)}\mathrm{d}s\Big] + \liminf_{k\to\infty} E\Big[\int_0^T\int_X |Q^{n_k}(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}t\Big]$$
$$\le E\Big[|G(T)|_H^2\Big] + 2E\Big[\int_0^T \langle X(s)-A(s,\Theta(s)),\Theta(s)\rangle \mathrm{d}s\Big]$$
$$+ 2E\Big[\int_0^T \langle A(s,\Theta(s)),Y(s)\rangle\mathrm{d}s\Big] + \lambda E\Big[\int_0^T |Y(s)-\Theta(s)|_H^2\mathrm{d}s\Big]$$
$$+ 2E\Big[\int_0^T \Big(b(s),Y(s)\Big)\mathrm{d}s\Big] \tag{3.27}$$

218 On the other hand, by (3.25), we have

$$E\Big[|Y_0|_H^2\Big] + E\Big[\int_0^T \|\bar{Z}(s)\|^2_{L_2(K,V)}\mathrm{d}s\Big] + E\Big[\int_0^T\int_X |Q(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}t\Big]$$
$$= E\Big[|G(T)|_H^2\Big] + 2E\Big[\int_0^T \langle X(s),Y(s)\rangle\mathrm{d}s\Big] + 2E\Big[\int_t^T \Big(b(s),Y(s)\Big)\mathrm{d}s\Big] \tag{3.28}$$

219 and by using (3.16)-(3.18), we get

$$E\Big[|Y_0|_H^2\Big] \le \liminf_{k\to\infty} E\Big[|Y^{n_k}(0)|_H^2\Big]$$
$$E\Big[\int_0^T \|\bar{Z}(s)\|^2_{L_2(K,V)}\mathrm{d}s\Big] \le \liminf_{k\to\infty} E\Big[\int_0^T \|\bar{Z}^{n_k}(s)\|^2_{L_2(K_n,V_n)}\mathrm{d}s\Big]$$
$$E\Big[\int_0^T\int_X |Q(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}t\Big] \le \liminf_{k\to\infty} E\Big[\int_0^T\int_X |Q^{n_k}(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}t\Big]$$

220 It then follows that

$$-2E\Big[\int_0^T \langle X(s)-A(s,\Theta(s)),Y(s)-\Theta(s)\rangle\mathrm{d}s\Big] + \lambda E\Big[\int_0^T |Y(s)-\Theta(s)|_H^2\mathrm{d}s\Big] \ge 0$$

221 Now, we set $\Theta = Y-\mu\Theta_1$ (for $\mu>0,\ \Theta_1\in M^2(0,T+\delta;H)\cap M^2(0,T+\delta;V)$), we get

$$-2E\Big[\int_0^T \langle X(s)-A(s,Y-\mu\Theta_1),\mu\Theta_1\rangle\mathrm{d}s\Big] + \lambda\mu^2 E\Big[\int_0^T |\Theta_1|_H^2\mathrm{d}s\Big] \ge 0 \tag{3.29}$$

We divide (3.29) by μ, take the limit when $\mu\to 0$ and use the hemicontinuity (a5) to obtain

$$-2E\Big[\int_0^T \langle X(s)-A(s,Y),\Theta_1\rangle\mathrm{d}s\Big] \ge 0,\ \text{for all } \Theta_1\in M^2(0,T+\delta;H)\cap M^p(0,T+\delta;V).$$

222 Hence $X=A(\cdot,Y)$. The proof is complete.

223 $\square$

Lemma 3.2. *Assume that the conditions of Theorem 3.1 hold. Moreover, assume that* $b(t,y,y_1,y_2,z,z_1,z_2,q,q_1,q_2,\omega) = b(t,z,z_1,z_2,q,q_1,q_2,\omega)$ *is independent of* y, y_1, y_2 *and that* $E\Big[\int_0^T |b(tz,z_1,z_2,q,q_1,q_2)|_H^2 dt\Big] < \infty$. *Then there exists a unique* $H \times L_2(K,H) \times L^2(\nu)$*-valued progressively measurable process* $(Y(t),Z(t),Q(t))$ *solution of equation (2.4) in* $M^2(0,T+\delta;H) \cap M^p(0,T+\delta;V) \times M^2(0,T+\delta;L_2(K,H)) \times M^2(0,T+\delta;L^2(\nu))$.

Proof of Lemma 3.2.

The uniqueness has already been shown.

Existence :

Set $Z^0(t) = 0$ and $Q^0(t,x) = 0$. For $n \geq 1$, define $(Y^n(t),Z^n(t),Q^n(t,x))$ to be the unique solution of the following BSPDEJ

$$\begin{cases} \mathrm{d}Y^n(t) & = -E\Big[b(t,Z^{n-1}(t),Z^{n-1}(t+\delta),Z_t^{n-1},Q^{n-1}(t,\cdot),Q^{n-1}(t+\delta,\cdot),Q_t^{n-1}(\cdot))\Big|\mathcal{F}_t\Big]\mathrm{d}t \\ & \qquad -A(t,Y^n(t))\mathrm{d}t + Z^n(t)\mathrm{d}B(t) + \int_X Q^n(t,\zeta)\widetilde{N}(\mathrm{d}t,\mathrm{d}\zeta), \;\; t\in[0,T] \\ Y^n(t) & = G(t), \;\; Z^n(t) = G_1(t), \;\; Q^n(t,\zeta) = G_2(t,\zeta), \;\; t\in[T,T+\delta], \end{cases} \tag{3.30}$$

The existence of solution of (3.30) is a consequence of Lemma 3.1. We shall show that $(Y^n(t),Z^n(t),Q^n(t,x))$ is a Cauchy sequence in $M^2(0,T+\delta;H)\cap M^p(0,T+\delta;V)\times M^2(0,T+\delta;L_2(K,H))\times M^2(0,T+\delta;L^2(\nu))$.

By the Itô formula, we have

$$\begin{aligned} 0 &= |G(T)-\bar{G}(T)|_H^2 = |Y^{n+1}(T)-Y^n(T)|_H^2 \\ &= |Y^{n+1}(t)-Y^n(t)|_H^2 - 2\int_t^T \langle A(s,Y^{n+1}(s)) - A(s,Y^n(s)), Y^{n+1}(s)-Y^n(s)\rangle \mathrm{d}s \\ &+ \int_t^T \|Z^{n+1}(s)-Z^n(s)\|_{L_2(K,H)}^2 \mathrm{d}s + \int_t^T\int_X |Q^{n+1}(s,\zeta)-Q^n(s,\zeta)|_H^2 N(\mathrm{d}s,\mathrm{d}\zeta) \\ &- 2\int_t^T \Big(E\Big[b^n(s)-b^{n-1}(s)\Big|\mathcal{F}_s\Big], Y^{n+1}(s)-Y^n(s)\Big)\mathrm{d}s \\ &+ 2\int_t^T \Big(Y^{n+1}(s)-Y^n(s), (Z^{n+1}(s)-Z^n(s))\mathrm{d}B(s)\Big) \\ &+ 2\int_t^T \Big(Y^{n+1}(s)-Y^n(s), (Q^{n+1}(s,\zeta)-Q^n(s,\zeta))\widetilde{N}(\mathrm{d}s,\mathrm{d}\zeta)\Big), \end{aligned} \tag{3.31}$$

where we have used the short hand notation

$$b^n(t) = b(t,Z^n(t),Z^n(t+\delta),Z_t^n,Q(t,\cdot),Q^n(t+\delta,\cdot),Q_t^n(\cdot))$$

and

$$b^{n-1}(t) = b(t,Z^{n-1}(t),Z^{n-1}(t+\delta),Z_t^{n-1},Q^{n-1}(t,\cdot),Q^{n-1}(t+\delta,\cdot),Q_t^{n-1}(\cdot))$$

We take the expectation, use the weak monotonicity argument (a4) of A and the Lipschitz condition on b to get

$$
\begin{aligned}
&E\Big[|Y^{n+1}(t)-Y^n(t)|_H^2\Big]+E\Big[\int_t^T \|Z^{n+1}(s)-Z^n(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big]\\
&+E\Big[\int_t^T\int_X |Q^{n+1}(s,\zeta)-Q^n(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]\\
&\le \lambda E\Big[\int_t^T |Y^{n+1}(s)-Y^n(s)|_H^2\mathrm{d}s\Big]+\frac{1}{\varepsilon}E\Big[\int_t^T |Y^{n+1}(s)-Y^n(s)|_H^2\mathrm{d}s\Big]\\
&+C_\varepsilon E\Big[\int_t^T \|Z^n(s)-Z^{n-1}(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big]+C_\varepsilon E\Big[\int_t^T \|Z_1^n(s)-Z_1^{n-1}(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big]\\
&+C_\varepsilon E\Big[\int_t^T \|Z_2^n(s)-Z_2^{n-1}(s)\|^2_{L^2(0,\delta;L_2(K,H))}\mathrm{d}s\Big]+C_\varepsilon E\Big[\int_t^T\int_X |Q^n(s,\zeta)-Q^{n-1}(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]\\
&+C_\varepsilon E\Big[\int_t^T\int_X |Q_1^n(s,\zeta)-Q_1^{n-1}(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]+C_\varepsilon E\Big[\int_t^T\int_X |Q_2^n(s,\zeta)-Q_2^{n-1}(s,\zeta)|^2_{L^2(0,\delta;H)}\nu(\mathrm{d}\zeta)\mathrm{d}s\Big].
\end{aligned}
$$

Since $Z^n(t)=Z^{n-1}(t)=G_1(t)$ for $t\in[T,T+\delta]$, we have from (3.3) and (3.4) that

$$
\begin{aligned}
&E\Big[\int_t^T \|Z_1^n(s)-Z_1^{n-1}(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big]\le E\Big[\int_t^T \|Z^{n+1}(s)-Z^n(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big],\\
&E\Big[\int_t^T \|Z_2^n(s)-Z_2^{n-1}(s)\|^2_{L^2(0,\delta;L_2(K,H))}\mathrm{d}s\Big]\le \delta E\Big[\int_t^T \|Z^{n+1}(s)-Z^n(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big].
\end{aligned}
$$

The same inequalities also hold for Q^n-Q^{n-1}. Choosing ε small enough (for e.g. such that $2C_\varepsilon+\delta C_\varepsilon\le \frac{1}{2}$) and using the preceding inequalities, we get

$$
\begin{aligned}
&E\Big[|Y^{n+1}(t)-Y^n(t)|_H^2\Big]+E\Big[\int_t^T \|Z^{n+1}(s)-Z^n(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big]\\
&+E\Big[\int_t^T\int_X |Q^{n+1}(s,\zeta)-Q^n(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]\\
&\le \lambda_\varepsilon E\Big[\int_t^T |Y^{n+1}(s)-Y^n(s)|_H^2\mathrm{d}s\Big]+\frac{1}{2}E\Big[\int_t^T \|Z^n(s)-Z^{n-1}(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big]\\
&+\frac{1}{2}E\Big[\int_t^T\int_X |Q^n(s,\zeta)-Q^{n-1}(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big], \qquad (3.32)
\end{aligned}
$$

where $\lambda_\varepsilon=\lambda+\dfrac{1}{\varepsilon}$. Hence

$$
\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t}\Big(e^{\lambda_\varepsilon t}E\Big[|Y^{n+1}(t)-Y^n(t)|_H^2\Big]\Big)+e^{\lambda_\varepsilon t}E\Big[\int_t^T\|Z^{n+1}(s)-Z^n(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big]\\
&+e^{\lambda_\varepsilon t}E\Big[\int_t^T\int_X|Q^{n+1}(s,\zeta)-Q^n(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]\\
&\leq\frac{1}{2}e^{\lambda_\varepsilon t}E\Big[\int_t^T\|Z^n(s)-Z^{n-1}(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big]\\
&+\frac{1}{2}e^{\lambda_\varepsilon t}E\Big[\int_t^T\int_X|Q^n(s,\zeta)-Q^{n-1}(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]. \qquad (3.33)
\end{aligned}
$$

Integrating (3.33) from 0 to T implies that

$$
\begin{aligned}
&E\Big[\int_0^T|Y^{n+1}(s)-Y^n(s)|_H^2\mathrm{d}s\Big]+\int_0^T e^{\lambda_\varepsilon t}E\Big[\int_t^T\|Z^{n+1}(s)-Z^n(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big]\mathrm{d}t\\
&+\int_0^T e^{\lambda_\varepsilon t}E\Big[\int_t^T\int_X|Q^{n+1}(s,\zeta)-Q^n(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]\mathrm{d}t\\
&\leq\frac{1}{2}\int_0^T e^{\lambda_\varepsilon t}E\Big[\int_t^T\|Z^n(s)-Z^{n-1}(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big]\mathrm{d}t\\
&+\frac{1}{2}\int_0^T e^{\lambda_\varepsilon t}E\Big[\int_t^T\int_X|Q^n(s,\zeta)-Q^{n-1}(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]\mathrm{d}t. \qquad (3.34)
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
&\int_0^T e^{\lambda_\varepsilon t}E\Big[\int_t^T\|Z^{n+1}(s)-Z^n(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big]\mathrm{d}t\\
&+\int_0^T e^{\lambda_\varepsilon t}E\Big[\int_t^T\int_X|Q^{n+1}(s,\zeta)-Q^n(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]\mathrm{d}t\\
&\leq\frac{1}{2}\int_0^T e^{\lambda_\varepsilon t}E\Big[\int_t^T\|Z^n(s)-Z^{n-1}(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big]\mathrm{d}t\\
&+\frac{1}{2}\int_0^T e^{\lambda_\varepsilon t}E\Big[\int_t^T\int_X|Q^n(s,\zeta)-Q^{n-1}(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]\mathrm{d}t. \qquad (3.35)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\int_0^T e^{\lambda_\varepsilon t}E\Big[\int_t^T\|Z^{n+1}(s)-Z^n(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big]\mathrm{d}t\\
&+\int_0^T e^{\lambda_\varepsilon t}E\Big[\int_t^T\int_X|Q^{n+1}(s,\zeta)-Q^n(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]\mathrm{d}t\\
&\leq(\frac{1}{2})^nC \qquad (3.36)
\end{aligned}
$$

for a suitable constant C. Thus, it follows form (3.35) that

$$E\Big[\int_0^T |Y^{n+1}(s)-Y^n(s)|_H^2\mathrm{d}s\Big] \le (\frac{1}{2})^n C. \tag{3.37}$$

Combining (3.37) and (3.32) gives

$$\begin{aligned}&E\Big[\int_t^T \|Z^{n+1}(s)-Z^n(s)\|_{L_2(K,H)}^2\mathrm{d}s\Big]+E\Big[\int_t^T\int_X |Q^{n+1}(s,\zeta)-Q^n(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]\\ &\le (\frac{1}{2})^n\lambda_\varepsilon C+\frac{1}{2}E\Big[\int_t^T \|Z^n(s)-Z^{n-1}(s)\|_{L_2(K,H)}^2\mathrm{d}s\Big]\\ &+\frac{1}{2}E\Big[\int_t^T\int_X |Q^n(s,\zeta)-Q^{n-1}(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big].\end{aligned} \tag{3.38}$$

Using (3.37) repeatedly yields

$$\begin{aligned}&E\Big[\int_t^T \|Z^{n+1}(s)-Z^n(s)\|_{L_2(K,H)}^2\mathrm{d}s\Big]+E\Big[\int_t^T\int_X |Q^{n+1}(s,\zeta)-Q^n(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big]\\ &\le (\frac{1}{2})^n n\lambda_\varepsilon C.\end{aligned}$$

for a suitable constant C. (3.37) and (3.38) imply that $(Y^n(t),Z^n(t),Q^n(t,x))$, $n\ge 1$, is a Cauchy sequence in $M^2(0,T+\delta;H)\times M^2(0,T+\delta;L_2(K,H))\times M^2(0,T+\delta;L^2(\nu))$. Therefore, there exists some progressively measurable process $(Y(t),Z(t),Q(t,x))$ such that $(Y^n(t),Z^n(t),Q^n(t,x))$ converges to $(Y(t),Z(t),Q(t,x))$ in $M^2(0,T+\delta;H)\times M^2(0,T+\delta;L_2(K,H))\times M^2(0,T+\delta;L^2(\nu))$.

Finally, letting $n\longrightarrow\infty$ in (3.30), we find that $(Y(t),Z(t),Q(t,x))$ satisfies

$$\begin{cases}\mathrm{d}Y(t) &= -E\Big[b(t,Z(t),Z(t+\delta),Z_t,Q(t,\cdot),Q(t+\delta,\cdot),Q_t(\cdot))\Big|\mathcal{F}_t\Big]\mathrm{d}t\\ &\qquad -A(t,Y(t))dt+Z(t)\mathrm{d}B(t)\ +\int_X Q(t,\zeta)\widetilde{N}(\mathrm{d}t,\mathrm{d}\zeta),\ \ t\in[0,T]\\ Y(t) &= G(t),\ Z(t)=G_1(t),\ Q(t,\zeta)=G_2(t,\zeta),\ \ t\in[T,T+\delta],\end{cases}$$

which means that $(Y(t),Z(t),Q(t,\zeta))$ is solution to the time-advanced BSDEJ (2.4). ☐

Proof of the general case

Let $Y^0(t)=0$. For $n\ge 1$, define $(Y^n(t),Z^n(t),Q^n(t,\zeta))$ to be the unique solution of the following BSPDEJ

$$\begin{cases}\mathrm{d}Y^n(t) &= -A(t,Y^n(t))dt-E\Big[b^{n-1}(t)\Big|\mathcal{F}_t\Big]\mathrm{d}t+Z^n(t)\mathrm{d}B(t)\\ &\qquad +\int_X Q^n(t,\zeta)\widetilde{N}(\mathrm{d}t,\mathrm{d}\zeta),\ \ t\in[0,T]\\ Y^n(t) &= G(t),\ Z^n(t)=G_1(t),\ Q^n(t,\zeta)=G_2(t,\zeta),\ \ t\in[T,T+\delta],\end{cases} \tag{3.39}$$

where

$$b^n(t)=b(t,Y^n(t),Y^n(t+\delta),Y_t^n,Z^n(t),Z^n(t+\delta),Z_t^n,Q^n(t,\cdot),Q^n(t+\delta,\cdot),Q_t^n(\cdot)).$$

The existence of a solution of (3.39) is a consequence of Lemma 3.2.

By using the same reasoning as in the proof of Lemma 3.2, it is possible to show that $(Y^n(t), Z^n(t), Q^n(t,x))$ is a Cauchy sequence in $M^2(0,T+\delta;H)\times M^2(0,T+\delta;L_2(K,H))\times M^2(0,T+\delta;L^2(\nu))$. Therefore $(Y^n(t), Z^n(t), Q^n(t,x))$ converges to some limit $(Y(t), Z(t), Q(t,x))$ which is the unique solution to the time-advanced BSDEJ (2.4). We omit the details. The proof of Theorem 3.1 is complete. □

3.2. **An estimate of the solution.** The following proposition gives an estimate of the solution of the time-advanced BSDEJ (2.4).

Proposition 3.3. *Assume that the condtions of Theorem 3.1 are satisfied. Then there exists a unique $H\times L_2(K,H)\times L^2(\nu)$-valued progressively measurable process $(Y(t), Z(t), Q(t))$ solution of equation (2.4) in $S^2(0,T+\delta;H)\cap M^p(0,T+\delta;V)\times M^2(0,T+\delta;L_2(K,H))\times M^2(0,T+\delta;L^2(\nu))$. Moreover, there exists a constant K depending on C in (b3), λ and α in (a1) and δ such that the solution $(Y(t), Z(t), Q(t))$ to the time-advanced BSDEJ (2.4) satisfies*

$$\begin{aligned}
&E\Big[\sup_{0\le t\le T}|Y(t)|_H^2+\int_0^T\Big\{\|Y(t)\|_V^p+\|Z(t)\|^2_{L_2(K,H)}+\int_X|Q(t,\zeta)|_H^2\nu(\mathrm{d}\zeta)\Big\}\mathrm{d}t\Big] \qquad (3.40)\\
&\le CE\Big[|G(T)|_H^2+\int_T^{T+\delta}\Big\{|G(t)|_H^2+\|G_1(t)\|^2_{L_2(K,H)}+\int_X|G_2(t,\zeta)|_H^2\nu(\mathrm{d}\zeta)\Big\}\mathrm{d}t\\
&\int_0^T|f(t)|\mathrm{d}t+\Big(\int_0^T|b(t,0,0,0,0,0,0,0,0,0)|_H\mathrm{d}t\Big)^2\Big]
\end{aligned}$$

Remark. Note that it follows from Theorem 3.1 that the solution $(Y(t), Z(t), Q(t))$ of equation (2.4) is in $M^2(0,T+\delta;H)\cap M^p(0,T+\delta;V)\times M^2(0,T+\delta;L_2(K,H))\times M^2(0,T+\delta;L^2(\nu))$. The estimate (3.40) enables us to conclude that $(Y(t), Z(t), Q(t))\in S^2(0,T+\delta;H)\cap M^p(0,T+\delta;V)\times M^2(0,T+\delta;L_2(K,H))\times M^2(0,T+\delta;L^2(\nu))$. We shall therefore only give the proof of estimate (3.40).

Proof. Here, we shall follow a similar proof as in [25] . By the Itô formula, we have

$$\begin{aligned}
&e^{\beta t}|Y(t)|_H^2+\int_t^T\beta e^{\beta s}|Y(s)|_H^2\mathrm{d}s+\int_t^T e^{\beta s}\|Z(s)\|^2_{L_2(K,H)}\mathrm{d}s+\int_t^T\int_X|Q(s,\zeta)|_H^2N(\mathrm{d}s,\mathrm{d}\zeta)\\
&=e^{\beta t}|G(T)|_H^2+2\int_t^T e^{\beta s}\langle A(s,Y(s)),Y(s)\rangle\mathrm{d}s+2\int_t^T e^{\beta s}\Big(E\Big[b(s)\Big|\mathcal{F}_s\Big],Y(s)\Big)\mathrm{d}s\\
&-2\int_t^T e^{\beta s}\Big(Y(s),(Z(s))\mathrm{d}B(s)\Big)-2\int_t^T e^{\beta s}\Big(Y(s),Q(s,\zeta)\widetilde{N}(\mathrm{d}s,\mathrm{d}\zeta)\Big), \qquad (3.41)
\end{aligned}$$

where we have used the short hand notation

$$b(t)=b(t,Y(t),Y(t+\delta),Y_t,Z(t),Z(t+\delta),Z_t,Q(t,\cdot),Q(t+\delta,\cdot),Q_t(\cdot))$$

Using coercivity, the Lipschitz condition and averaging, we get

$$
\begin{aligned}
&e^{\beta t}|Y(t)|_H^2+\int_t^T \beta e^{\beta s}|Y(s)|_H^2\mathrm{d}s+\int_t^T e^{\beta s}\|Z(s)\|_{L_2(K,H)}^2\mathrm{d}s+\int_t^T\int_X e^{\beta s}|Q(s,\zeta)|_H^2 N(\mathrm{d}s,\mathrm{d}\zeta)\\
&\le e^{\beta t}|G(T)|_H^2-2\int_t^T e^{\beta s}\Big(Y(s),(Z(s))\mathrm{d}B(s)\Big)-2\int_t^T e^{\beta s}\Big(Y(s),\int_X Q(s,\zeta)\widetilde{N}(\mathrm{d}s,\mathrm{d}\zeta)\Big)\\
&+\lambda\int_t^T e^{\beta s}|Y(s)|_H^2\mathrm{d}s-\alpha\int_t^T e^{\beta s}\|Y(s)\|_V^p\mathrm{d}s+\int_t^T f(s)\mathrm{d}s\\
&+2C\int_t^T e^{\beta s}|Y(s)|_H^2\mathrm{d}s+\frac{1}{\varepsilon_1}C\int_t^T e^{\beta s}|Y(s)|_H^2\mathrm{d}s+\varepsilon_1 C\int_t^T E\Big[e^{\beta s}|Y_1(s)|_H^2\Big|\mathcal{F}_s\Big]\mathrm{d}s \qquad (3.42)\\
&+\frac{1}{\varepsilon_2}C\int_t^T e^{\beta s}|Y(s)|_H^2\mathrm{d}s+\varepsilon_2 C\int_t^T e^{\beta s}E\Big[\|Y_2(s)\|_{L^2(0,\delta;H)}^2\Big|\mathcal{F}_s\Big]\mathrm{d}s\\
&+\frac{1}{\varepsilon_3}C\int_t^T e^{\beta s}|Y(s)|_H^2\mathrm{d}s+\varepsilon_3 C\int_t^T e^{\beta s}\|Z(s)\|_{L_2(K,H)}^2\mathrm{d}s\\
&+\frac{1}{\varepsilon_4}C\int_t^T e^{\beta s}|Y(s)|_H^2\mathrm{d}s+\varepsilon_4 C\int_t^T e^{\beta s}E\Big[\|Z_1(s)\|_{L_2(K,H)}^2\Big|\mathcal{F}_s\Big]\mathrm{d}s\\
&+\frac{1}{\varepsilon_5}C\int_t^T e^{\beta s}|Y(s)|_H^2\mathrm{d}s+\varepsilon_5 C\int_t^T e^{\beta s}E\Big[\|Z_2(s)\|_{L^2(0,\delta;L_2(K,H))}^2\Big|\mathcal{F}_s\Big]\mathrm{d}s\\
&+\frac{1}{\varepsilon_6}C\int_t^T e^{\beta s}|Y(s)|_H^2\mathrm{d}s+\varepsilon_6 C\int_t^T\int_X e^{\beta s}|Q(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\\
&+\frac{1}{\varepsilon_7}C\int_t^T e^{\beta s}|Y(s)|_H^2\mathrm{d}s+\varepsilon_7 C\int_t^T\int_X e^{\beta s}E\Big[|Q_1(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\Big|\mathcal{F}_s\Big]\mathrm{d}s\\
&+\frac{1}{\varepsilon_8}C\int_t^T e^{\beta s}|Y(s)|_H^2\mathrm{d}s+\varepsilon_8 C\int_t^T\int_X e^{\beta s}E\Big[|Q_2(s,\zeta)|_{L^2(0,\delta;H)}^2\nu(\mathrm{d}\zeta)\Big|\mathcal{F}_s\Big]\mathrm{d}s.\\
&+2\int_t^T e^{\beta s}\Big(E\Big[b(s,0,0,0,0,0,0,0,0,0)\Big|\mathcal{F}_s\Big],Y(s)\Big)\mathrm{d}s
\end{aligned}
$$

It also follows from (3.3) and (3.4) that

$$
\begin{aligned}
E\Big[\int_t^T|Y_1(s)|_H^2\mathrm{d}s\Big]&=E\Big[\int_t^T|Y(s+\delta)|_H^2\mathrm{d}s\Big]\\
&\le E\Big[\int_t^T|Y(s)|_H^2\mathrm{d}s\Big]+E\Big[\int_T^{T+\delta}|G(s)|_H^2\mathrm{d}s\Big], \qquad (3.43)\\
E\Big[\int_t^T|Y_2(s)|_{L^2(0,\delta;H)}^2\mathrm{d}s\Big]&\le\delta\Big(E\Big[\int_t^T|Y(s)|_H^2\mathrm{d}s\Big]+E\Big[\int_T^{T+\delta}|G(s)|_H^2\mathrm{d}s\Big]\Big), \qquad (3.44)
\end{aligned}
$$

and similarly for Z and Q.

We take conditional expectation with respect to $\mathcal{F}_t$ and use (3.43) and (3.44) to get

$$
\begin{aligned}
&e^{\beta t}|Y(t)|_H^2 + E\Big[\int_t^T \beta e^{\beta s}|Y(s)|_H^2 \mathrm{d}s\Big|\mathcal{F}_t\Big] + E\Big[\int_t^T e^{\beta s}\|Z(s)\|_{L_2(K,H)}^2 \mathrm{d}s\Big|\mathcal{F}_t\Big]\\
&+ E\Big[\int_t^T\int_X e^{\beta s}|Q(s,\zeta)|_H^2 N(\mathrm{d}s,\mathrm{d}\zeta)\Big|\mathcal{F}_t\Big] + \alpha E\Big[\int_t^T e^{\beta s}\|Y(s)\|_V^p \mathrm{d}s\Big|\mathcal{F}_t\Big]\\
&\leq E\Big[e^{\beta t}|G(T)|_H^2\Big|\mathcal{F}_t\Big] + C^1_{\lambda,\varepsilon,\delta}E\Big[\int_t^T e^{\beta s}|Y(s)|_H^2 \mathrm{d}s\Big|\mathcal{F}_t\Big] + E\Big[\int_t^T f(s)\mathrm{d}s\Big|\mathcal{F}_t\Big]\\
&+ C^2_{\varepsilon,\delta}E\Big[\int_t^T e^{\beta s}\|Z(s)\|_{L_2(K,H)}^2 \mathrm{d}s\Big|\mathcal{F}_t\Big] + C^3_{\varepsilon,\delta}E\Big[\int_t^T\int_X e^{\beta s}|Q(s,\zeta)|_H^2 \nu(\mathrm{d}\zeta)\mathrm{d}s\Big|\mathcal{F}_t\Big] \qquad (3.45)\\
&+ 2E\Big[\int_t^T e^{\beta s}\Big(b(s,0,0,0,0,0,0,0,0,0),Y(s)\Big)\mathrm{d}s\Big|\mathcal{F}_t\Big] + C^4_{\varepsilon,\delta}E\Big[\int_T^{T+\delta} e^{\beta s}|G(s)|_H^2 \mathrm{d}s\Big|\mathcal{F}_t\Big]\\
&+ C^5_{\varepsilon,\delta}E\Big[\int_T^{T+\delta} e^{\beta s}\|G_1(s)\|_{L_2(K,H)}^2 \mathrm{d}s\Big|\mathcal{F}_t\Big] + C^6_{\varepsilon,\delta}E\Big[\int_T^{T+\delta}\int_X e^{\beta s}|G_2(s,\zeta)|_H^2 \nu(\mathrm{d}\zeta)\mathrm{d}s\Big|\mathcal{F}_t\Big]
\end{aligned}
$$

where

$$
\begin{aligned}
&C^1_{\lambda,\varepsilon,\delta} = \lambda + C(2 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3} + \frac{1}{\varepsilon_4} + \frac{1}{\varepsilon_5} + \frac{1}{\varepsilon_6} + \frac{1}{\varepsilon_7} + \frac{1}{\varepsilon_8} + \varepsilon_1 + \varepsilon_2\delta),\\
&C^2_{\varepsilon,\delta} = C(\varepsilon_3 + \varepsilon_4 + \delta\varepsilon_5),\;\; C^3_{\varepsilon,\delta} = C(\varepsilon_6 + \varepsilon_7 + \delta\varepsilon_8)\\
&C^4_{\varepsilon,\delta} = C(\varepsilon_1 + \delta\varepsilon_2),\;\; C^5_{\varepsilon,\delta} = C(\varepsilon_4 + \delta\varepsilon_5),\;\; C^6_{\varepsilon,\delta} = C(\varepsilon_7 + \delta\varepsilon_8)
\end{aligned}
$$

Choose $\beta = C^1_{\lambda,\varepsilon,\delta}$ to obtain

$$
\begin{aligned}
&e^{\beta t}|Y(t)|_H^2 + \alpha E\Big[\int_t^T e^{\beta s}\|Y(s)\|_V^p \mathrm{d}s\Big|\mathcal{F}_t\Big] + E\Big[\int_t^T e^{\beta s}\|Z(s)\|_{L_2(K,H)}^2 \mathrm{d}s\Big|\mathcal{F}_t\Big]\\
&+ E\Big[\int_t^T\int_X e^{\beta s}|Q(s,\zeta)|_H^2 N(\mathrm{d}s,\mathrm{d}\zeta)\Big|\mathcal{F}_t\Big] \qquad (3.46)\\
&\leq C^2_{\varepsilon,\delta}E\Big[\int_t^T e^{\beta s}\|Z_1(s)\|_{L_2(K,H)}^2 \mathrm{d}s\Big|\mathcal{F}_t\Big] + C^3_{\varepsilon,\delta}E\Big[\int_t^T\int_X e^{\beta s}|Q(s,\zeta)|_H^2 \nu(\mathrm{d}\zeta)\mathrm{d}s\Big|\mathcal{F}_t\Big]\\
&+ 2E\Big[\int_t^T e^{\beta s}\Big(b(s,0,0,0,0,0,0,0,0,0),Y(s)\Big)\mathrm{d}s\Big|\mathcal{F}_t\Big] + E\Big[\int_t^T |f(s)|\mathrm{d}s\Big|\mathcal{F}_t\Big]\\
&+ C^4_{\varepsilon,\delta}E\Big[\int_T^{T+\delta} e^{\beta s}|G(s)|_H^2 \mathrm{d}s\Big|\mathcal{F}_t\Big] + C^5_{\varepsilon,\delta}E\Big[\int_T^{T+\delta} e^{\beta s}\|G_1(s)\|_{L_2(K,H)}^2 \mathrm{d}s\Big|\mathcal{F}_t\Big]\\
&+ C^6_{\varepsilon,\delta}E\Big[\int_T^{T+\delta}\int_X e^{\beta s}|G_2(s,\zeta)|_H^2 \nu(\mathrm{d}\zeta)\mathrm{d}s\Big|\mathcal{F}_t\Big].
\end{aligned}
$$

Choose $\varepsilon_i,\ i = 3,\dots,8$ such that $C^2_{\varepsilon,\delta} < 1,\ C^3_{\varepsilon,\delta} < 1$; then we obtain

$$
\begin{aligned}
&E\Big[\int_t^T e^{\beta s}\|Y(s)\|_V^p \mathrm{d}s\Big|\mathcal{F}_t\Big] + E\Big[\int_t^T e^{\beta s}\|Z(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big|\mathcal{F}_t\Big]\\
&+E\Big[\int_t^T\int_X e^{\beta s}|Q(s,\zeta)|_H^2 N(\mathrm{d}s,\mathrm{d}\zeta)\Big|\mathcal{F}_t\Big]\\
&\le C_{\varepsilon,\delta,\alpha}\Big(E\Big[\int_t^T e^{\beta s}\Big(b(s,0,0,0,0,0,0,0,0,0),Y(s)\Big)\mathrm{d}s\Big|\mathcal{F}_t\Big] + E\Big[\int_t^T |f(s)|\mathrm{d}s\Big|\mathcal{F}_t\Big]\\
&+E\Big[\int_T^{T+\delta} e^{\beta s}|G(s)|_H^2\mathrm{d}s\Big|\mathcal{F}_t\Big] + E\Big[\int_T^{T+\delta} e^{\beta s}\|G_1(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big|\mathcal{F}_t\Big]\\
&+E\Big[\int_T^{T+\delta}\int_X e^{\beta s}|G_2(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big|\mathcal{F}_t\Big]\Big) \qquad (3.47)
\end{aligned}
$$

for an appropriate constant $C_{\varepsilon,\delta,\alpha}$. For $t\in[r,T]$, we have

$$
\int_t^T e^{\beta s}\Big(Y(s),(Z(s))\mathrm{d}B(s)\Big) \le \Big|\int_r^T e^{\beta s}\Big(Y(s),(Z(s))\mathrm{d}B(s)\Big)\Big| + \Big|\int_r^t e^{\beta s}\Big(Y(s),(Z(s))\mathrm{d}B(s)\Big)\Big|
$$

and by the Burkholder–Davis–Gundy inequality, we get

$$
\begin{aligned}
&E\Big[\sup_{t\le r\le T}\Big|\int_r^T e^{\beta s}\Big(Y(s),(Z(s))\mathrm{d}B(s)\Big)\Big|\Big|\mathcal{F}_r\Big]\\
&\le C_{1,1}E\Big[\sup_{r\le s\le T} e^{\beta s}|Y(s)|_H^2\Big|\mathcal{F}_r\Big] + C_{1,2}E\Big[\int_r^T e^{\beta s}\|Z(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big|\mathcal{F}_r\Big] \qquad (3.48)
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&E\Big[\sup_{r\le t\le T}\Big|\int_r^T e^{\beta s}\Big(Y(s),Q(s,\zeta)\widetilde{N}(\mathrm{d}s,\mathrm{d}\zeta)\Big)\Big|\Big|\mathcal{F}_r\Big]\\
&\le C_{2,1}E\Big[\sup_{r\le s\le T} e^{\beta s}|Y(s)|_H^2\Big|\mathcal{F}_r\Big] + C_{2,2}E\Big[\int_r^T\int_X e^{\beta s}|Q(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big|\mathcal{F}_r\Big]. \qquad (3.49)
\end{aligned}
$$

We take the supremum in (3.45),and take the conditional expectation with respect to $\mathcal{F}_u$. Then by using (3.48) and (3.49) we obtain

$$
\begin{aligned}
&E\Big[\sup_{u\le t\le T} e^{\beta t}|Y(t)|_H^2\Big|\mathcal{F}_u\Big]\\
&\le E\Big[e^{\beta t}|G(T)|_H^2\Big|\mathcal{F}_u\Big] + 2E\Big[\sup_{u\le t\le T}\Big|\int_t^T e^{\beta s}\Big(Y(s),(Z(s))\mathrm{d}B(s)\Big)\Big|\Big|\mathcal{F}_u\Big]\\
&+ 2E\Big[\sup_{u\le t\le T}\Big|\int_t^T e^{\beta s}\Big(Y(s),Q(s,\zeta)\widetilde{N}(\mathrm{d}s,\mathrm{d}\zeta)\Big)\Big|\Big|\mathcal{F}_u\Big]\\
&+ C^1_{\lambda,\varepsilon,\delta}E\Big[\int_u^T e^{\beta s}|Y(s)|_H^2\mathrm{d}s\Big|\mathcal{F}_u\Big] + C^2_{\varepsilon,\delta}E\Big[\int_u^T e^{\beta s}\|Z_1(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big|\mathcal{F}_u\Big]\\
&+ C^3_{\varepsilon,\delta}E\Big[\int_u^T\int_X e^{\beta s}|Q(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s - 2\int_u^T e^{\beta s}\Big(b(s,0,0,0,0,0,0,0,0,0),Y(s)\Big)\mathrm{d}s\Big|\mathcal{F}_u\Big]\\
&+ C^4_{\varepsilon,\delta}E\Big[\int_T^{T+\delta} e^{\beta s}|G(s)|_H^2\mathrm{d}s\Big|\mathcal{F}_u\Big] + C^5_{\varepsilon,\delta}E\Big[\int_T^{T+\delta} e^{\beta s}\|G_1(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big|\mathcal{F}_u\Big]\\
&+ C^6_{\varepsilon,\delta}E\Big[\int_T^{T+\delta}\int_X e^{\beta s}|G_2(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big|\mathcal{F}_u\Big] + E\Big[\int_u^T|f(s)|\mathrm{d}s\Big|\mathcal{F}_u\Big]\\
&\le E\Big[e^{\beta t}|G(T)|_H^2\Big|\mathcal{F}_u\Big] + C_{1,1}E\Big[\sup_{u\le t\le T} e^{\beta t}|Y(t)|_H^2\Big|\mathcal{F}_u\Big] + C_{1,2}E\Big[\int_u^T e^{\beta s}\|Z(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big|\mathcal{F}_u\Big]\\
&+ C_{2,1}E\Big[\sup_{u\le t\le T} e^{\beta t}|Y(t)|_H^2\Big|\mathcal{F}_u\Big] + C_{2,2}E\Big[\int_u^T\int_X e^{\beta s}|Q(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big|\mathcal{F}_u\Big]\\
&+ C^{1,1}_{\varepsilon,\delta}E\Big[\int_u^T e^{\beta s}|Y(s)|_H^2\mathrm{d}s\Big|\mathcal{F}_u\Big] + C^2_{\varepsilon,\delta}E\Big[\int_u^T e^{\beta s}\|Z_1(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big|\mathcal{F}_u\Big]\\
&+ C^3_{\varepsilon,\delta}E\Big[\int_u^T\int_X e^{\beta s}|Q(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s + 2\int_u^T e^{\beta s}|b(s,0,0,0,0,0,0,0,0,0)|_H|Y(s)|_H\mathrm{d}s\Big|\mathcal{F}_u\Big]\\
&+ C^4_{\varepsilon,\delta}E\Big[\int_T^{T+\delta} e^{\beta s}|G(s)|_H^2\mathrm{d}s\Big|\mathcal{F}_u\Big] + C^5_{\varepsilon,\delta}E\Big[\int_T^{T+\delta} e^{\beta s}\|G_1(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big|\mathcal{F}_u\Big]\\
&+ C^6_{\varepsilon,\delta}E\Big[\int_T^{T+\delta}\int_X e^{\beta s}|G_2(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big|\mathcal{F}_u\Big] + E\Big[\int_u^T|f(s)|\mathrm{d}s\Big|\mathcal{F}_u\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\le E\Big[e^{\beta t}|G(T)|_H^2\Big|\mathcal{F}_u\Big] + (C_{1,1}+C_{2,1}+C^{1,1}_{\varepsilon,\delta}+C_{3,1})E\Big[\sup_{u\le t\le T} e^{\beta t}|Y(t)|_H^2\Big|\mathcal{F}_u\Big]\\
&+ (C_{1,2}+C^{2}_{\varepsilon,\delta})E\Big[\int_u^T e^{\beta s}\|Z(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big|\mathcal{F}_u\Big]\\
&+ (C_{2,2}+C^{3}_{\varepsilon,\delta})E\Big[\int_u^T\int_X e^{\beta s}|Q(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big|\mathcal{F}_u\Big]\\
&+ C_{3,2}E\Big[\Big(\int_u^T e^{\frac{1}{2}\beta s}|b(s,0,0,0,0,0,0,0,0,0)|_H\mathrm{d}s\Big)^2\Big|\mathcal{F}_u\Big]\\
&+ C^{4}_{\varepsilon,\delta}E\Big[\int_T^{T+\delta} e^{\beta s}|G(s)|_H^2\mathrm{d}s\Big|\mathcal{F}_u\Big] + C^{5}_{\varepsilon,\delta}E\Big[\int_T^{T+\delta} e^{\beta s}\|G_1(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big|\mathcal{F}_u\Big]\\
&+ C^{6}_{\varepsilon,\delta}E\Big[\int_T^{T+\delta}\int_X e^{\beta s}|G_2(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big|\mathcal{F}_u\Big] + E\Big[\int_u^T |f(s)|\mathrm{d}s\Big|\mathcal{F}_u\Big]
\end{aligned}
$$

Let K denote an appropriate constant depending only on T, C, δ, α. Then from the preceding inequalities and (3.47), we obtain

$$
\begin{aligned}
&E\Big[\sup_{u\le t\le T} e^{\beta t}|Y(t)|_H^2\Big|\mathcal{F}_u\Big]\\
&\le KE\Big[e^{\beta t}|G(T)|_H^2\Big|\mathcal{F}_u\Big] + (C_{1,1}+C_{2,1}+C^{1,1}_{\varepsilon,\delta}+C_{3,1}+C_{4,1})E\Big[\sup_{u\le t\le T} e^{\beta t}|Y(t)|_H^2\Big|\mathcal{F}_u\Big]\\
&+ KE\Big[\Big(\int_u^T e^{\frac{1}{2}\beta s}|b(s,0,0,0,0,0,0,0,0,0)|_H\mathrm{d}s\Big)^2\Big|\mathcal{F}_u\Big]\\
&+ KE\Big[\int_T^{T+\delta} e^{\beta s}|G(s)|_H^2\mathrm{d}s\Big|\mathcal{F}_u\Big] + KE\Big[\int_T^{T+\delta} e^{\beta s}\|G_1(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big|\mathcal{F}_u\Big]\\
&+ KE\Big[\int_T^{T+\delta}\int_X e^{\beta s}|G_2(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big|\mathcal{F}_u\Big] + E\Big[\int_u^T |f(s)|\mathrm{d}s\Big|\mathcal{F}_u\Big].
\end{aligned}
$$

Finally, choose $C_{1,1}$, $C_{2,1}$, $C^{1,1}_{\varepsilon,\delta}$, $C_{3,1}$ and $C_{4,1}$ such that $C_{1,1}+C_{2,1}+C^{1,1}_{\varepsilon,\delta}+C_{3,1}+C_{4,1}<1$. Then

$$
\begin{aligned}
&E\Big[\sup_{u\le t\le T} e^{\beta t}|Y(t)|_H^2\Big|\mathcal{F}_u\Big] &\quad(3.50)\\
&\le K\Big(E\Big[|G(T)|_H^2\Big|\mathcal{F}_u\Big]+E\Big[\Big(\int_u^T|b(s,0,0,0,0,0,0,0,0,0)|_H\mathrm{d}s\Big)^2\Big|\mathcal{F}_u\Big]\\
&+E\Big[\int_T^{T+\delta}|G(s)|_H^2\mathrm{d}s\Big|\mathcal{F}_u\Big]+E\Big[\int_T^{T+\delta}\|G_1(s)\|^2_{L_2(K,H)}\mathrm{d}s\Big|\mathcal{F}_u\Big]\\
&+E\Big[\int_T^{T+\delta}\int_X|G_2(s,\zeta)|_H^2\nu(\mathrm{d}\zeta)\mathrm{d}s\Big|\mathcal{F}_u\Big]+E\Big[\int_u^T|f(s)|\mathrm{d}s\Big|\mathcal{F}_u\Big]\Big). &\quad(3.51)
\end{aligned}
$$

A combination of (3.47) and (3.50) leads to the desired estimate for an appropriate K depending on T, C, δ, α.

□

Remark. Let us remark that the results obtained are still valid if we assume that V is a separable reflexive Banach space dense in the Hilbert space H.

4. Application to time-advanced backward stochastic porous-medium equations

In this section, we apply the results to study a time-advanced backward stochastic porous-medium equation.

Let $\mathcal{O}$ be an open bounded subset of $\mathbb{R}^n$, $p \ge 2$ and define

$$V := L^p(\mathcal{O}),\ \ V^* := (L^p(\mathcal{O}))^*,\ \ H := (H_0^{1,2}(\mathcal{O}))^*.$$

We shall consider $H_0^{1,2}(\mathcal{O})$ with the scalar product

$$\langle u, v\rangle := \int \langle \nabla u(x), \nabla v(x)\rangle \mathrm{d}s,\ u, v \in H_0^{1,2}(\mathcal{O})$$

Since $H_0^{1,2}(\mathcal{O}) \subset L^{\frac{p}{p-1}}(\mathcal{O})$ it follows that

$$L^p(\mathcal{O}) = (L^{\frac{p}{p-1}}(\mathcal{O}))^* \subset (H_0^{1,2}(\mathcal{O}))^* = H$$

continuously and densely. In order to identify H with its dual H^*, we define the Riez isomorphism $(-\Delta)^{-1} : H \to H^* = H_0^{1,2}(\mathcal{O})$ by

$$\langle u, \cdot\rangle_H = \langle(-\Delta)^{-1}u, \cdot\rangle_{H^*,H} \text{ for every } u \in H$$

With this, we have $H \equiv H^*$, whence (see [27])

$$V = L^p(\mathcal{O}) \hookrightarrow H \equiv H^* \hookrightarrow (L^p(\mathcal{O}))^*.$$

Let Φ be a real measurable function from $\mathbb{R}$ into $\mathbb{R}$ satisfying the following properties:

P1: Φ is continuous.

P2: For all $s, t \in \mathbb{R}$

$$(t - s)(\Phi(t) - \Phi(s)) \ge 0.$$

P3: There exists $p \in [2,\infty),\ \mu \in (0,\infty),\ \sigma \in [0,\infty)$ such that for all $s \in \mathbb{R}$

$$s\Phi(s) \geq \mu|s|^p - \sigma.$$

P4: There exist $\beta_1, \beta_2 \in [0,\infty)$such that for all $s \in \mathbb{R}$

$$|\Phi(s)| \leq \beta_3 + \beta_4|s|^{p-1},$$

where p is given as in **P3**.

It follows that

$$\Phi(u) \in L^{\frac{p}{p-1}}(\mathcal{O}) \text{ for all} u \in L^p(\mathcal{O})$$

Define the *porous medium operator* $A : V = L^p(\mathcal{O}) \to V^* = (L^p(\mathcal{O}))^*$ by

$$A(u) := \Delta\Phi(u),\ \ u \in L^p(\mathcal{O}).$$

For $u, v \in L^p(\mathcal{O})$ we have

$$\langle u, A(v)\rangle = -\int v(x)\Phi(u(x))\,\mathrm{d}x$$

We shall consider the following time-advance BSPMEJs

$$\begin{aligned} dY(t,x) &= -\Delta\Phi(Y(t,x))dt - E\Big[b(t,x)\Big|\mathcal{F}_t\Big]\mathrm{d}t + Z(t,x)\,\mathrm{d}B(t) \\ &\quad + \int_X Q(t,x,\zeta)\widetilde{N}(\mathrm{d}t,\mathrm{d}\zeta),\ \ (t,x) \in [0,T]\times\mathcal{O} \end{aligned} \tag{4.1}$$

$$Y(t,x) = G(t,x),\ \ G(\cdot,x) \in S^2(T,T+\delta;H) \cap M^p(T,T+\delta;V) \tag{4.2}$$

$$Z(t,x) = G_1(t,x),\ \ G_1(\cdot,x) \in M^2(T,T+\delta;L_2(K,H)) \tag{4.3}$$

$$Q(t,x,\zeta) = G_2(t,x,\zeta),\ \ G_2(\cdot,x) \in M^2(T,T+\delta;L^2(\nu)) \tag{4.4}$$

where $b(t,x) = b(t,x,Y(t,x),Y(t+\delta,x),Y_t(x),Z(t,x),Z(t+\delta,x),Z_t,Q(t,x,\cdot),Q(t+\delta,\cdot),Q_t(\cdot))$.

It can be verified (see [27]) that the operator A satisfies assumptions (a1)-(a5). Thus, we can apply the result of Theorem 3.1.

Note that if we choose $\Phi(u) = |u|^{p-2}u$ in the definition of A then we obtain the classical porous medium operator.

Acknowledgements

The author thanks Priyanka Parbhoo for helpful comments and suggestions.

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